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S-parameter Modeling of Passive Components

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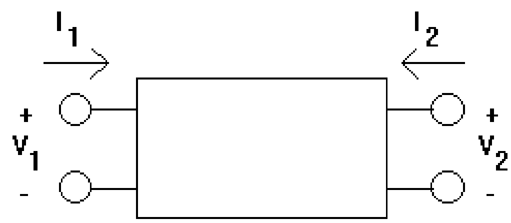
Abstract --Modern network analysis depends heavily upon the use of scattering parameters (S-parameters) to specify the behavior of two-port devices. There are many sophisticated treatments of the subject of scattering matrices and network analysis but the level of sophistication creates difficulty if a reader desires a concise reference for occasional use. This paper is a basic discussion of the relationship between S-parameters and circuit theory, including transmission line devices. The analysis will be limited to the 'ABCD' matrix method. The document was originally presented in a Mathcad 6.0 format. The file is available from the author.

Two-port ABCD analysis



Consider the 'black box' shown to the right. We can measure voltage and current at port 1 (V_1 and I_1) and also at port 2 (V_2 and I_2). Note that the current is always measured as if it was flowing *into* the black box. In principle, we can fix any two of the voltages and currents and measure the remaining two to get information about what is inside the box. Since there are four variables and since we will independently control two of them, there are six different ways to get information. (4!/(2!2!) = 6). We will be using only two ... the ABCD matrix, and the scattering matrix.

If we fix the voltage and current at port one and measure the voltage and current at port two, we get a pair of equations:

$$V_1 = A \cdot V_2 - B \cdot I_2$$
$$I_1 = C \cdot V_2 - D \cdot I_2$$

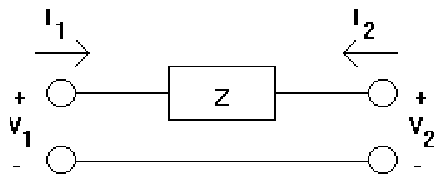
At first glance, this representation might seem so counterintuitive that it couldn't possibly be useful. But, it turns out to be very useful because it is a 'multipurpose' analysis tool, like the Swiss army knife of network theory. 'A,' 'B,' 'C,' and 'D' are sometimes called the *general circuit parameters* of the two port network. 'A' and 'D' are dimensionless ratios, while 'B' and 'C' have the respective dimensions of Ohms and Seimens.

The general circuit parameters may be represented in matrix notation:
(Remember -- multiply matrices row by column.)

Of course, this is the ABCD matrix ....

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \qquad \text{eq. 1}$$

Probably the best way to 'get the hang of it' is to consider some examples:

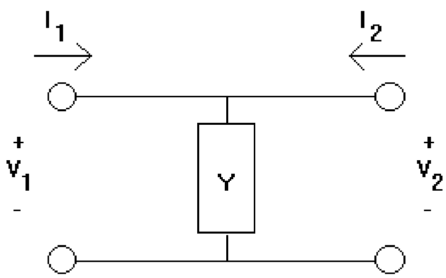


This is a series impedance. It has the dimension, Ohms. We know from Kirchoff's laws that:

$$V_1 = V_2 - Z \cdot I_2$$
$$I_1 = -I_2$$

We can see by inspection that the matrix to the right correctly describes the situation.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \quad \text{eq. 2.}$$

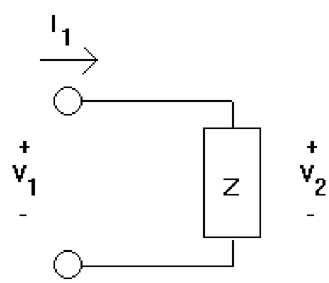


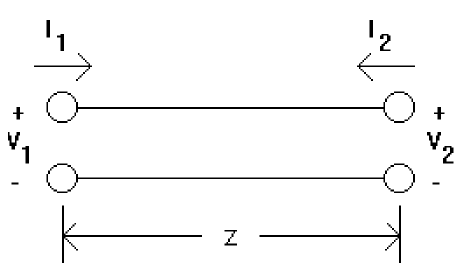
This is a shunt admittance. It has the dimension, Seimens. We can use the same process as before to show that its matrix is:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \quad \text{eq. 3.}$$

This is a terminating impedance. It is actually a one port network, but it will come in handy later. Of course, $V_1 = V_2$ and $I_2 = 0$. We will state formally that the terminating impedance's matrix is:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{Z} & 0 \end{pmatrix} \quad \text{eq. 4.}$$





This is a transmission line section. From transmission line theory, we know that when port two is connected to a terminating impedance (Z_2) the impedance at port one is:

$$Z_1 = Z_C \cdot \frac{Z_2 + Z_C \cdot \tanh(\gamma \cdot z)}{Z_C + Z_2 \cdot \tanh(\gamma \cdot z)}$$

In transmission line theory, Z_C is called the characteristic impedance, gamma is the propagation factor and z is the propagation distance. I claim that the ABCD matrix of a transmission line section is ...

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} \cosh(\gamma\cdot z) & Z_C\cdot\sinh(\gamma\cdot z) \\ \dfrac{\sinh(\gamma\cdot z)}{Z_C} & \cosh(\gamma\cdot z) \end{bmatrix} \quad \text{eq. 5.}$$

... which we will check by trying to reproduce the 'known' answer from transmission line theory. (This is where you will begin to see the power of the ABCD matrix for network calculations.)

When you want to know the ABCD matrix of a combination of elements, you simply multiply the separate ABCD matrices of the elements. Thus for the terminated transmission line:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} \cosh(\gamma\cdot z) & Z_C\cdot\sinh(\gamma\cdot z) \\ \dfrac{\sinh(\gamma\cdot z)}{Z_C} & \cosh(\gamma\cdot z) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \dfrac{1}{Z_2} & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\left(\cosh(\gamma\cdot z)\cdot Z_2 + Z_C\cdot\sinh(\gamma\cdot z)\right)}{Z_2} & 0 \\ \dfrac{\left(\sinh(\gamma\cdot z)\cdot Z_2 + \cosh(\gamma\cdot z)\cdot Z_C\right)}{\left(Z_C\cdot Z_2\right)} & 0 \end{bmatrix}$$

Now, recall that:

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \quad \text{eq. 1}$$

We substitute the ABCD matrix for the terminated transmission line ...

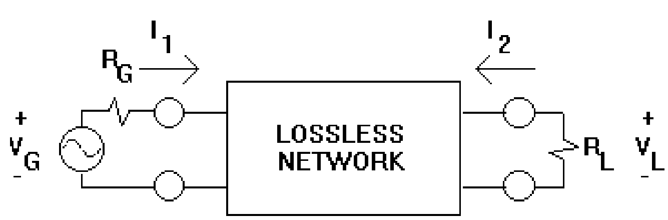
$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{bmatrix} \dfrac{\left(\cosh(\gamma\cdot z)\cdot Z_2 + Z_C\cdot\sinh(\gamma\cdot z)\right)}{Z_2} & 0 \\ \dfrac{\left(\sinh(\gamma\cdot z)\cdot Z_2 + \cosh(\gamma\cdot z)\cdot Z_C\right)}{\left(Z_C\cdot Z_2\right)} & 0 \end{bmatrix} \cdot \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{bmatrix} \dfrac{\left(\cosh(\gamma\cdot z)\cdot Z_2 + Z_C\cdot\sinh(\gamma\cdot z)\right)}{Z_2}\cdot V_2 \\ \dfrac{\left(\sinh(\gamma\cdot z)\cdot Z_2 + \cosh(\gamma\cdot z)\cdot Z_C\right)}{\left(Z_C\cdot Z_2\right)}\cdot V_2 \end{bmatrix}$$

... and invoke Ohm's law (Z = V/I) ...

$$Z_1 = \frac{\left(\cosh(\gamma\cdot z)\cdot Z_2 + Z_C\cdot\sinh(\gamma\cdot z)\right)}{\left(\sinh(\gamma\cdot z)\cdot Z_2 + \cosh(\gamma\cdot z)\cdot Z_C\right)}\cdot Z_C = Z_C\cdot\frac{Z_2 + Z_C\cdot\tanh(\gamma\cdot z)}{Z_C + Z_2\cdot\tanh(\gamma\cdot z)}$$

... which agrees with the expression above.

Power transfer, voltage transfer



Here is a lossless network connecting a generator and a load resistance. Under DC conditions, we know that the condition for maximum power transfer is $R_G = R_L$ so that $V_L = 1/2\ V_G$. Under AC conditions, we know this is not true -- We could connect any R_L to any R_G, as long as the connecting means includes a lossless transformer, and still achieve maximum power transfer. Space permitting, we could show that ...

$$V_L(\text{available}) = \frac{1}{2}\cdot\sqrt{\frac{R_L}{R_G}}\cdot V_G \quad \text{eq. 6.}$$

... is the AC version of the DC power transfer theorem.

Of course, if the network connecting the generator to the load has losses, V_L will be less than V_L(available). The ratio, t = V_L/V_L(available), would be a good measurement of the efficiency of the connecting network. Thus ...

$$t = \frac{V_L}{V_L(\text{available})} = 2\cdot\sqrt{\frac{R_G}{R_L}}\cdot\frac{V_L}{V_G} \quad \text{eq. 7.}$$

Now we can compute t, the voltage transfer function, of any network placed between RG and RL by simply calculating the ratio, V_L/V_G, and applying that to eq. 7. Just use eq. 1 and a little bit of Ohm's law. (We don't need to do anything with I_1.)

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \left[\begin{pmatrix} 1 & R_G \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]\cdot\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}$$

This is just eq. 1, with the combination of the series generator resistance and the unknown ABCD matrix multiplied together inside brackets.

$$\begin{pmatrix} V_G \\ I_G \end{pmatrix} = \begin{bmatrix} V_L\cdot A + V_L\cdot R_G\cdot C - \frac{V_L}{R_L}\cdot B - \frac{V_L}{R_L}\cdot R_G\cdot D \\ C\cdot V_L - D\cdot I_L \end{bmatrix}$$

Here, we have defined the port voltages and currents and carried out the matrix multiplications.

$$\frac{V_L}{V_G} = \frac{R_L}{A\cdot R_L + B + C\cdot R_G\cdot R_L + D\cdot R_G}$$

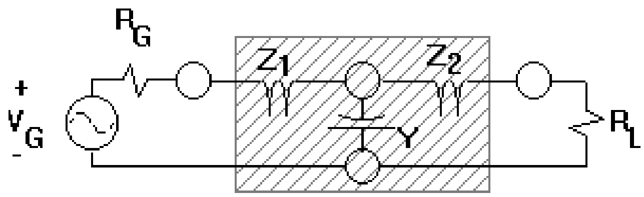
We only need the first row. Here it is, appropriately rearranged.

Here is the voltage transfer function, t, defined in terms of terminating impedances (R_G and R_L) and an unknown ABCD matrix.

$$t = 2\cdot\sqrt{\frac{R_G}{R_L}}\cdot\frac{V_L}{V_G} = \frac{2\cdot\sqrt{R_G\cdot R_L}}{A\cdot R_L + B + C\cdot R_G\cdot R_L + D\cdot R_G} \quad \text{eq. 8.}$$

Voltage reflection

Here is an interesting network consisting of a generator, two inductors (Z_1 and Z_2), a capacitor (Y) and a load. We already know how to calculate the impedance 'seen' by I_1 by combining the appropriately selected ABCD matrices. To make life simpler, let's rearrange the units of measurement to make R_L = 1.



This process, called normalization, is really no different than converting inches to feet or minutes to hours. We will just express all impedances and admitances as fractions of R_L. Of course, now all we have to do is to divide row one by row two to find the 'terminating impedance,' Z_t.

$$\begin{pmatrix} 1 & Z_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & Z_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 + Z_1 \cdot Y + Z_2 + Z_2 \cdot Z_1 \cdot Y + Z_1 & 0 \\ Y + Y \cdot Z_2 + 1 & 0 \end{pmatrix}$$

$$Z_t = \frac{1 + Z_1 \cdot Y + Z_2 + Z_2 \cdot Z_1 \cdot Y + Z_1}{Y + Y \cdot Z_2 + 1}$$

Now, let us think about the circuit and what happens to it when we excite it with an AC source. The elements inside the box are all reactive, so the electric energy we put into them at one point in the cycle will be returned to the circuit at a later time. When this energy is returned to the circuit, it will take the path of least resistance, which means it could actually flow back into the generator. This phenomenon is called reflection. We will now calculate a *reflection coefficient* based on voltages and currents in the circuit.

First, we know that if we divide a circuit into two parts, the voltages of the two parts have to be equal at the dividing line. (This is Kirchoff's Voltage Law.) We will choose as our dividing line the beginning of the circuit which is causing the reflection. Thus:

$$V_i = V_t - V_r$$

Second, we know that at any given point in the circuit, the sum of currents flowing into the point has to be balanced by the sum of the currents flowing out of the point. (Kirchoff's Current Law) Thus:

$$I_i = I_t + I_r$$

We can express current in terms of voltage and impedance. (Ohm's Law) Thus:

$$I_i = \frac{V_i}{R_G} \qquad I_t = \frac{V_t}{Z_t} \qquad I_r = \frac{V_r}{R_G}$$

Using algebra, we can form the ratio of reflected to incident voltage:

$$\frac{V_r}{V_i} = \Gamma = \frac{Z_t - R_G}{Z_t + R_G} \qquad \text{eq. 9.}$$

And also the ratio of transmitted to incident voltage:

$$\frac{V_t}{V_i} = \tau = \frac{2 \cdot Z_t}{Z_t + R_G} \qquad \text{eq. 10.}$$

Energy conservation

We know that the energy, and thus the power, incident on Z_t must be either transmitted to Z_t or reflected. We can write this knowledge in the form of an equation:

$$P_i = P_t + P_r \qquad \text{or} \qquad \frac{P_t}{P_i} + \frac{P_r}{P_i} = 1$$

Since power is associated with the square of voltage, let's take a stab at summing the squares of gamma and tau –

$$\left(\frac{Z_t - R_G}{Z_t + R_G}\right)^2 + \left(\frac{2 \cdot Z_t}{Z_t + R_G}\right)^2 = \frac{\left(Z_t^2 - 2 \cdot Z_t \cdot R_G + R_G^2\right) + 4 \cdot Z_t^2}{Z_t^2 + 2 \cdot Z_t \cdot R_G + R_G^2} \neq 1$$

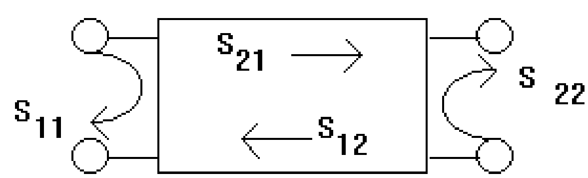
What happened? It looks as if we should have multiplied the second term on the left by (R_G/Z_t) ...

$$\left(\frac{Z_t - R_G}{Z_t + R_G}\right)^2 + \frac{R_G}{Z_t} \cdot \left(\frac{2 \cdot Z_t}{Z_t + R_G}\right)^2 = \frac{\left(Z_t^2 - 2 \cdot Z_t \cdot R_G + R_G^2\right) + 4 \cdot Z_t \cdot R_G}{Z_t^2 + 2 \cdot Z_t \cdot R_G + R_G^2} = 1$$

This is exactly how the voltage transfer function, t, is related to tau!

$$t = \sqrt{\frac{R_G}{R_L}} \cdot \tau \qquad \text{eq. 11.}$$

Scattering parameters



We are almost ready to talk about S-parameters. First, here is some background information.

At high frequencies, 'voltage' becomes an abstract concept because it can't be directly measured! All we can really measure is incident, transmitted, and reflected power. Further, it is a practical impossibility to build a microwave detector that can be considered an open circuit. Lastly, most circuits are designed so that their impedances are all normalized to either 50 or 75 Ohms. Thus, when you want to specify or test the microwave performance of a network, S-parameters become the natural choice.

S-parameters have two indices. The first index signifies the destination port, while the second index identifies the source port. Thus, 'S_{21}' is scatter toward port two from port one and 'S_{11}' is scatter toward port one from port one. We have already talked about these two, but now we write them in their most general form.

$$S_{11} = \Gamma_1 = \frac{A \cdot Z_L + B - C \cdot \overline{Z_S} \cdot Z_L - D \cdot \overline{Z_S}}{A \cdot Z_L + B + C \cdot Z_S \cdot Z_L + D \cdot Z_S} \qquad \text{eq. 12.}$$

$$S_{21} = t_{21} = \frac{2 \cdot \sqrt{R_S \cdot R_L}}{A \cdot Z_L + B + C \cdot Z_S \cdot Z_L + D \cdot Z_S} \qquad \text{eq. 13.}$$

Here, the overbar symbolizes 'complex conjugate.' Z_S and Z_L are the (complex) source and load impedances. R_S and R_L are the real parts of the source and load impedances.
We could calculate S_{12} and S_{22} by looking from Z_L toward Z_S. Here they are in their most general form.

$$S_{12} = t_{12} = \frac{2 \cdot (A \cdot D - B \cdot C) \cdot \sqrt{R_S \cdot R_L}}{A \cdot Z_L + B + C \cdot Z_S \cdot Z_L + D \cdot Z_S} \qquad \text{eq. 14.}$$

$$S_{22} = \Gamma_2 = \frac{-A \cdot \overline{Z_L} + B - C \cdot Z_S \cdot \overline{Z_L} + D \cdot Z_S}{A \cdot Z_L + B + C \cdot Z_S \cdot Z_L + D \cdot Z_S} \qquad \text{eq. 15.}$$

When $Z_S = Z_L$ and both are strictly real, we shall say that $Z_S = Z_L = R_0$. Now everything gets much simpler, because we can also normalize by R_0.

$$S_{11} = \frac{A + B - C - D}{A + B + C + D} \qquad S_{12} = \frac{2 \cdot (A \cdot D - B \cdot C)}{A + B + C + D}$$

$$S_{21} = \frac{2}{A + B + C + D} \qquad S_{22} = \frac{-A + B - C + D}{A + B + C + D}$$

Two final comments –

When S-parameters are plotted, they are usually seen in dB, thus:

$$\left|S_{ij}\right| \cdot dB = 20 \cdot \log_{10}\left(\left|S_{ij}\right|\right)$$

When they are tabulated for computer aided design, they are usually *not* seen in matrix order, frequency, magnitude, angle format (sometimes called S2P) thus:

$$\text{frequency} \quad \left|S_{11}\right| \quad \arg\left(S_{11}\right) \quad \left|S_{21}\right| \quad \arg\left(S_{21}\right) \quad \left|S_{12}\right| \quad \arg\left(S_{12}\right) \quad \left|S_{22}\right| \quad \arg\left(S_{22}\right)$$

References

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